mistaken ; for fired and inspired by this very lecture, Harry Hart went home and discovered the five-bar perfect parallel motion now known by his name, where for Peaucellier's cell of six bars is substituted Hart's contraparallelogram of four.

ON A METHOD TO CONSTRUCT INTRANSITIVE SUBSTITUTION GROUPS.

By DR. G. A. MILLER.

An intransitive group contains two or more transitive constituents and may be constructed by establishing some isomorphism between these constituent groups. If it contains more than two transitive constituents we may combine them in any way into two constituents and construct the group by establishing some isomorphism between these constituents, where at least one of them is an intransitive group. For example, we can find all the possible groups of degree 11 whose transitive constituents are of degrees 4, 4, 3 by establishing isomorphisms between the intransitive groups of degree 8 which contain two transitive constituents of degree 4 and the transitive groups of degree three. We can also find all these groups by establishing isomorphisms between the intransitive groups of degree 7 whose systems of intransitivity are 4, 3 and the transitive groups of degree 4.

Sometimes we can observe some important group properties from the notation by which an intransive group is represented. For instance, if the group is not simply isomorphic to some one of its transitive constituents it must contain at least two self-conjugate subgroups, differing from identity, that do not have any common operator besides identity. Conversely, if a group contains two such self-conjugate subgroups it may be represented as an intransitive group which is not simply isomorphic to any one of its transitive constituents.

While the method of establishing some isomorphism between two constituent groups seems to be the best general method to construct intransitive groups yet it is sometimes desirable to employ others. To illustrate one of these we shall employ it to find all the possible intransitive groups of degree 10 that contain five systems of intransitivity.

The average number of elements in all the substitutions of such a group is $10-5=5$* and a positive substitution must be of degree 4 or 8, while a negative substitution must be of degree 2, 6, or 10. There is evidently only one such group of order 2. If a group of order 4 contains only positive substitutions it must contain x substitutions of degree 4 and y of degree 8, where $x+y=3$. Hence

*Frobenius, *Crelle*, vol. 101, p. 287; cf. Miller, *Bulletin of the American Mathematical Society*, vol. 2, 1895, p. 75.

$$4x+8(3-x)=4.5\,; \quad x=1, \quad y=2.$$

Since the substitution of degree 4 and one of the substitutions of degree 8 must generate the required group of degree 10 they can have only one transposition in common. Hence there is only one positive group of order 4.

If a group of order 4 contains negative substitutions it must involve one positive substitution besides identity. If this is of degree 4 the two negative substitutions must involve $20-4=16$ elements. Hence one of them must be of degree 6 and the other of degree 10. Since the one of degree 6 and that of degree 4 must generate the entire group there is only one such group. If the positive substitution is of degree 8, the two negative substitutions must involve $20-8=12$ elements. There is clearly one group whose negative substitutions are composed of one transposition and one substitution of degree 10, and one whose negative substitutions contain three transpositions. Hence there are four groups of order 4,—one of these contains only positive substitutions while the other three contain negative substitutions.

If a group of order 8 contains only positive substitutions it must contain x substitutions of degree 4 and y of degree 8, where $x+y=7$. Hence

$$4x+8(7-y)=8.5\;; \quad x=4.$$

The substitutions of degree 4 must therefore generate the entire group. Since at least two of them must have a common transposition such a group must contain a subgroup of order 4 and degree 6. There is only one such group because the remaining generating substitution of degree four must involve the 4 elements that are not found in the given subgroup of order 4.

We proceed to consider the groups of order 8 that include negative substitutions. The positive subgroup of order 4 must be of degree 6, 8, or 10. If it is of degree 6 its substitutions must involve 12 elements. The four negative substitutions of the group must therefore involve $40-12=28$ elements. Since there must be one substitution of degree 10 there can be only one such group. If the given subgroup is of degree 8 the four negative substitutions of the group must include $40-16=24$ elements. There is clearly one group that contains a single transposition and another that contains four substitutions of degree 6.

Finally, if the given positive subgroup is of degree 10 the four negative substitutions must involve $40-20=20$ elements. Hence at least one of these substitutions must be a transposition. If this transposition is found in the substitution of degree 4 the group will contain two transpositions; if it is not contained in this substitution the group will contain only one transposition. We have now considered all the possible cases and found that there are six groups of order 8,—one of these is positive while the other five contain negative substitutions.

The largest possible group of degree 10 that contains five systems of intransitivity is of order $2^5=32$. There is evidently only one group of this order. From this it follows directly that there is only one positive group of order 16 that contains the given systems of intransitivity. The three groups of this order that

contain negative substitutions may be found in exactly the same manner as those of order 8. Hence there are just 16 groups of degree 10 that contain five systems of intransitivity,—1 of order 2, 4 of order 4, 6 of order 8, 4 of order 16, and 1 of order 32. All those that have the same order represent the same abstract group.

Cornell University, November, 1898.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society ; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from May Number.]

PROPOSITION XXXVI. *If any straight XF* (Fig. 44) *makes an acute angle with any ordinate LF [of the equisdistantial], the point X does not fall without the cavity of the curve, unless previously XF has cut the curve in some point O.*

PROOF. It is certain that the point X may be assumed in XF so near to the point F, that the join LX previously cuts the curve in some point S: otherwise XF either does not fall wholly without the cavity of the curve, and so we have our assertion ; or so it does not make with FL an acute angle, rather it would be to suppose that XL comes together with LF in one same straight.

Accordingly from the point S let fall to the base AB the perpendicular SP. This will be (from P. 34) equal to LF.

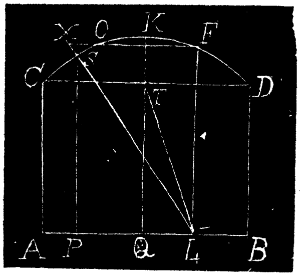

Fig. 44.

But SP is (from Eu. I, 18) less than LS.

Therefore also LF is less than LS, and therefore much less than LX. Hence in triangle LXF the angle at the point X will be acute, because less (from Eu. I, 18) than the angle LFX supposed acute.

Now let fall to FX the perpendicular LT. This falls (because of Eu. I, 17) toward the parts of each acute angle. Wherefore the point T will lie between the points X and F. Then from the point T let fall to the base AB the perpendicular TQ.

LF (because of the right angle at T) will be greater than LT, and this (because of the right angle at Q) will be greater than QT. Therefore LF will be far greater than QT. But hence, if in QT produced QK is taken equal to LF, the point K (from P. 34) will pertain to the present curve, and therefore the point T falls within the cavity of this curve.

Therefore the straight FT, which cuts two straights QK and LT in T, can